Mathematics 222B Lecture 6 Notes

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February 3, 2022

1 Hölder Spaces, Bounded Mean Oscillation, and Compact Operators

1.1 Hölder spaces

Let's continue our discussion of Sobolev inequalities. We want to know: What does $\|u\|_{W^{1,p}}$ say about u when $p \geq d$? We proved a lemma:

Lemma 1.1. *Suppose $u \in C^\infty(\mathbb{R}^d)$ with $d \geq 2$. Then*

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u(x) - u(z)| \, dz \leq C \int_{B_r(x)} \frac{|Du(z)|}{|z-x|^{d-1}} \, dz$$

From this lemma, we saw the following theorem:

Theorem 1.1. *Let $u \in C^\infty(\mathbb{R}^d)$ with $d \geq 2$, and let $x, y \in B_R$. Then*

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|Du\|_{L^p(B_R)},$$

where $\alpha = 1 - \frac{d}{p}$.

We want to rephrase this as an inequality for $u \in W^{1,p}(U)$. To do this, we need a space that has a regularity property relating to the theorem above.

Definition 1.1. Let $u \in C(I)$. The **Hölder seminorm** of order α is

$$[u]_{C^\alpha(U)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

By a **seminorm**, we mean that $[\cdot]_{C^\alpha(U)}$ satisfies all the properties of a norm except the property that $[u]_{C^\alpha(U)} = 0 \implies u = 0$. Instead, this implies that u is constant. Here is how we make it into a norm

Definition 1.2. The **Hölder norm** of order α is

$$\|u\|_{C^\alpha(U)} = [u]_{C^\alpha(U)} + \|u\|_{L^\infty}.$$

The **Hölder space** of order α is

$$C^\alpha(U) = \{u \in C(U) : \|u\|_{C^\alpha} < \infty\}.$$

Theorem 1.2 (Morrey's inequality[1])**.** *Let $d \geq 2$, let $p > d$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. If $u \in W^{1,p}(U)$, then $u \in C^\alpha(U)$ with $\alpha = 1 - \frac{d}{p}$. Moreover,*

$$\|u\|_{C^\alpha(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Proof. By extension and density theorems, it suffices to consider $u \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subseteq V$, where V is a bounded, open set with $V \supseteq \overline{U}$ (chosen independently of u). By the previous theorem,

$$[u]_{C^\alpha(V)} \leq C\|u\|_{W^{1,p}}.$$

So all that remains is to bound $\|u\|_{L^\infty}$ in terms of $\|u\|_{W^{1,p}}$. For this purpose, we will again use the lemma to approximate u by its average. Let $x \in V$. Then

$$\begin{aligned}
\left| u(x) - \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, dz \right| &\leq \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} u(x) - u(z) \, dz \right| \\
&\leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(x) - u(z)| \, dz \\
&\leq C \int_{B_r(x)} \frac{|Du(z)|}{|z - x|^{d-1}} \, dz \\
&\leq C r^\alpha \|Du\|_{L^p(B_r(x))}.
\end{aligned}$$

Take $r = 1$. Then

$$\begin{aligned}
|u(x)| &\leq C \underbrace{\left| \int_{B_r(x)} u \, dz \right|}_{\leq \int_{B_1(x)} |u| \, dz \leq C\|u\|_{L^p(B_1(0))}} + C\|Du\|_{L^p} \\
&\leq C(\|u\|_{L^p} + \|Du\|_{L^p}).
\end{aligned}$$

□

1.2 Bounded mean oscillation

When $p = d$, $W^{1,d}$ does not embed into L^∞.

[1] This is sometimes called Morey's embedding.

Example 1.1. For $d = 2$, let $U = B_1(0)$, and consider

$$u = \log\log\left(10 + \frac{1}{|x|}\right).$$

A useful substitute for the above failure involves the space of bounded mean oscillation (BMO).

Definition 1.3. Let $u \in L^1_{\text{loc}}(U)$. The **BMO seminorm** is

$$[u]_{\text{BMO}} = \sup_{B_r(x_0)\subseteq U} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left| u(z) - \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u \right| dz.$$

Theorem 1.3. *Let $d \geq 2$, $U \subseteq \mathbb{R}^d$, and $u \in W^{1,d}(\mathbb{R}^d)$. Then $[u]_{\text{BMO}} < \infty$, and*

$$[u]_{\text{BMO}} \leq C\|Du\|_{L^d}.$$

Remark 1.1. As an exercise, you can show that $L^\infty \subsetneq \text{BMO}$. The function $u = \mathbb{1}_{B_1(0)} \log|x|$ shows that these spaces are nor equal.

Proof. Assume $u \in C^\infty(\mathbb{R}^d)$. We want to show that

$$[u]_{\text{BMO}} \leq C\|Du\|_{L^d}.$$

Fix $B_r(x)$. We want to show that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \left| u(z) - \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy \right| dz \leq C\|Du\|_{L^d}.$$

with some fixed constant C. We can rewrite the left hand side to get

$$\begin{aligned}
&\frac{1}{|B_r(x)|}\int_{B_r(x)} \left| \frac{1}{|B_r(x)|}u(z)\,dy - \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy \right| dz \\
&\qquad\qquad \leq \frac{1}{|B_r(x)|^2}\int_{B_r(x)}\int_{B_r(x)} |u(z) - u(y)|\,dy\,dz
\end{aligned}$$

Since $B_r(x) \subseteq B_{2r}(y)$,

$$\leq \frac{1}{|B_r(x)|^2}\int_{B_r(x)}\int_{B_{2r}(y)} |u(z) - u(y)|\,dy\,dz$$

Using the lemma,

$$\leq \frac{1}{|B_r(x)|^2}\int_{B_r(x)} \underbrace{\int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}}\,dz}_{F(y)}\,dy$$

This is a convolution, so you might be tempted to use Young's inequality: $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$, where $1 \leq p \leq q \leq r \leq \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. However, this barely fails, since $\frac{1}{|z-x|^{d-1}} \notin L^q$. Instead, we use the following theorem:

Theorem 1.4 (Hardy-Littlewood)**.** *Let $u \in L^1_{\text{loc}}$, and define*

$$\mathcal{M}u(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|.$$

(Note that $|\mathcal{M}u| \leq \|u\|_{L^\infty}$). For $1 < p \leq \infty$,

$$\|\mathcal{M}u\|_{L^p} \leq C\|u\|_{L^p}.$$

Whenever you are faced with something that is hard to understand, it is a good idea to decompose the region into pieces where the function is mostly constant. The power function $|y|^\alpha$ has the property that if $2^{k-1} \leq |y|, |y'| \leq 2^k$, then $|y|^\alpha \simeq |y'|^\alpha$. For our problem, write $A_k = \{2^{k-1} \leq |z-y| \leq 2^k\}$, so

$$\begin{aligned}
\int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}}\, dz &\leq C \sum_{2^k \leq 2r+c} \int_{A_k} \frac{1}{(2^k)^{d-1}} |Du(z)|\, dz \\
&\leq C \sum_{2^k \leq 2cr} \frac{1}{(2^k)^{d-1}} \int_{B_{2^k}(y)} |Du(z)|\, dz \\
&\leq C \sum_{2^k \leq 2cr} 2^k \mathcal{M}(|Du|)(y).
\end{aligned}$$

It now suffices to bound

$$\left\| \sum_{2^k \leq 2cr} 2^k \mathcal{M}(|Du|)(y) \right\|_{L^1} \leq Cr \|\mathcal{M}|Du|\|_{L^d} \|1\|_{L^{\frac{d}{d-r}}(B_r(x))}$$

Using the theorem,

$$\leq Cr^d \|Du\|_{L^d}. \qquad \square$$

1.3 Compact operators and embeddings

We will discuss two more topics involving Sobolev spaces:

1. Compactness of Sobolev embedding
2. Poincaré-type inequalities (how to get information about u from $\|Du\|_{L^p}$ given some extra condition for normalizing the function).

Let's set up the discussion for the first topic.

Definition 1.4. Let X, Y be normed spaces, and let $T : X \to Y$ be linear. We say that T is a **compact operator** if $T(B_X)$, the image of the unit ball in X, is compact in Y. Equivalently, we may require that for all bounded $\{x_n\} \subseteq X$, $\{Tx_n\}$ has a convergent subsequence.

Definition 1.5. Suppose that we have an embedding (i.e. a bounded, linear, injective map) $\iota : X \to Y$. We say the embedding $X \subseteq Y$ is **compact** if ι is compact.

We are interested in writing something like this: $W^{1,p}(U) \subseteq L^q(U)$. If we think of $W^{1,p}(U)$ as a subspace of functions, then this embedding will be compact.

What is the basic compactness theorem in the setting of function spaces? We will use the Arzelà-Ascoli theorem:

Theorem 1.5 (Arzelà-Ascoli)**.** *Let K be a compact set and $\mathcal{A} \subseteq C(K)$. Suppose that*

1. *$\mathcal{A}$ is **locally bounded**, i.e. for any $x \in K$, there is an $M(x)$ such that for all $f \in \mathcal{A}$, $|f(x)| \leq M(x)$.*
2. *$\mathcal{A}$ is **equicontinuous**, i.e. for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{A}$,*

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon, \qquad \forall x, y \in K.$$

Then $\mathcal{A}$ is compact.